The Gluon Dyson-Schwinger equation of lattice Landau gauge
(DSE of a link variable)

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Motivation

This study was motivated by the outcome of the investigation of *Mader, Schaden, Zwanziger* and *Alkofer* [EPJC(2014)74:2881]

- Investigated the QEoM (DSE) of the gauge boson

$$\delta_{\sigma\mu}\delta^{ab} = \left\langle A^a_\sigma(y)\frac{\partial S}{\partial A^a_\mu(x)}\right\rangle_{\mathcal{FT}} = -\left\langle A^a_\sigma(y)\,\partial_\nu F^b_{\nu\mu}(x)\right\rangle_{\mathcal{FT}} - \left\langle A^a_\sigma(y)\,\tilde{j}^b_\mu(x)\right\rangle_{\mathcal{FT}}$$

  in various gauges and models with a BRST symmetry
  (Lin./gen. covariant, maximal Abelian, non-covariant Coulomb, Gribov-Zwanziger)

- Local current $\tilde{j}^a_\mu(x) = j^a_\mu(x) + s\xi^a_\mu(x)$ differs from conserved Noether current j by a BRST-exact term (physically equivalent if BRST unbroken)

- Expressed the Kugo-Ojima confinement criterion in terms of the saturation of this equation at $p \to 0$ (corollary to KO-criterion)

- term which saturates *rhs* for $p \to 0$ depends on phase of a gauge theory
  Higgs phase: physical states contribute to transverse equation
  Confining: saturation entirely due to unphysical *dofs*

Motivation

Can we find / define similar on the lattice?

- Yet unclear
- DSE of a link variable in Landau gauge has never been considered before (...moving onto new ground)

First steps (this talk)

- Have to define the Gluon DSE on the lattice in terms of link variables, i.e., the lattice analog of $\delta_{\mu\nu}\delta^{ab}\delta(x-y) = \left\langle A^a_\mu(x)\frac{\delta S}{\delta A^b_\nu(y)}\right\rangle$
- Fundamental degrees of freedom (dof.) are link variables
- Calculate KO-function $u(p^2)$ to see if it saturates *lhs* also on the lattice
- Focus on Landau gauge and Wilson gauge action (pure YM theory)

Gauge-fixing on the lattice

Landau gauge

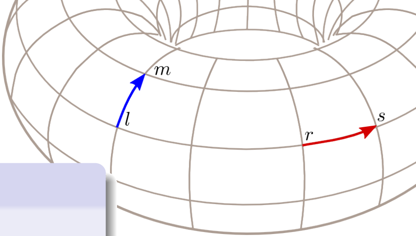
- Gauge configurations $U = \{U_{ij}\}$ are generated using gauge-invariant measure $d\mu_W = D[U]e^{S_W[U]}$ (heatbath, HMC for unquenched, ...)
- Gauge-fixing performed subsequently via iterative algorithm (over-relaxation, Fourier-accelerated gauge-fixing, simulated-annealing, ...)
- Find a gauge-transformation $U_{ij} \to U^g_{ij} = g_i U_{ij} g^\dagger_j$ which minimizes the Morse potential (gauge functional)

$$V[U^g] = -\sum_{ij} \mathrm{Tr}\, U^g_{ij} \qquad (i,j) = \text{neighboring sites at } (x, x \pm \hat{\mu})$$

- Minima not unique: many different minima per U (Gribov copies)
- All Gribov copies fulfill lattice Landau gauge (LLG) condition

$$0 = f^a_i = \mathfrak{Re} \sum_j \mathrm{Tr}\, t^a U^g_{ij} \qquad \forall (i,a), \quad t^a = -t^{\dagger a} = \text{generator of SU(n)}$$

Variation of one link variable



Left variation $\underline{\boldsymbol{\delta}}^b_{lm}$

$$\underline{\boldsymbol{\delta}}^b_{lm} U_{rs} := \underbrace{t^b U_{lm}\ \delta_{rl}\delta_{sm} - U_{ml} t^b\ \delta_{rm}\delta_{sl}}_{\text{left variation}} + \underbrace{\theta_r U_{rs} - U_{rs}\theta_s}_{\text{gauge transformation}}$$

- Left-variation of the link variable U_{lm}: $U_{lm} \to (1+h)U_{lm}, \qquad (h = h^b t^b)$
- Infinitesimal gauge transformation θ_i (returns configuration to LLG)

Alternative form

$$\underline{\boldsymbol{\delta}}^b_{x\mu} U_{y\nu} := t^b U_{x\mu}\ \delta_{yx}\delta_{y+\hat{\nu},x+\hat{\mu}} - U^\dagger_{x\mu} t^b\ \delta_{y,x+\hat{\mu}}\delta_{y+\hat{\nu},x} + \theta_y U_{y\nu} - U_{y\nu}\theta_{y+\hat{\nu}}$$

where we identified $(l,m) = (x, x+\hat{\mu})$ and $(r,s) = (y, y+\hat{\nu})$.

Landau gauge condition and the Faddeev-Popov matrix

Variation leaves Landau Gauge condition fulfilled

$$0 = \underline{\delta}^b_{lm} f^a_i = \mathfrak{Re}\,\mathrm{Tr}[t^a t^b U_{lm}(\delta_{il} - \delta_{im}) + \sum_j t^a(\theta_i U_{ij} - U_{ij}\theta_j)]$$

- $\{\theta_i, i \in \Lambda\}$ are (anti-hermitian) traceless $n \times n$ matrices
- Ensure that the configuration remains in LLG
- Components of $\theta_i = \sum_c t^c \theta^{c;b}_{i;lm}$ solve the linear system

$$\sum_{c,j} M^{bc}_{ij} \theta^{c;a}_{j;lm} = \mathcal{U}^{ba}_{lm}(\delta_{il} - \delta_{im})$$

Faddeev-Popov (FP) matrix

$$M^{ab}_{ij} := \mathcal{U}^{ba}_{ij} - \delta_{ij} \sum_k \mathcal{U}^{ab}_{ik} \quad \text{with} \quad \mathcal{U}^{ab}_{ij} = \mathcal{U}^{ba}_{ji} := \mathfrak{Re}\,\mathrm{Tr}\, t^a t^b U_{ij}$$

The Faddeev-Popov trick

Functional

$$\mathcal{D}_\alpha^{-1}[U] := \int \prod_i dg_i \, e^{-\alpha V[U^g]}$$

- Gauge-invariant by construction $\mathcal{D}_\alpha^{-1}[U] = \mathcal{D}_\alpha^{-1}[U^g]$
- For $\alpha \to \infty$ (Landau gauge) reduces to sum over all low-lying Minima U_k of U

$$\mathcal{D}_\alpha^{-1}[U] \xrightarrow{\alpha\to\infty} \sum_k e^{-\alpha V[U_k]} \left(\det M[U_k]\right)^{-1/2}$$

Inserting the unity

$$1 = \int \prod_i dg_i \, e^{-\alpha V[U^g]} \mathcal{D}_\alpha[U]$$

in the gauge-invariant lattice measure gives gauge-fixed model with the positive measure

$$d\mu_\alpha = D[U] \, \mathcal{D}_\alpha[U] \; e^{S_W[U] - \alpha V[U]}$$

Vacuum expectation values

Expectation value of gauge-variant observable in Landau gauge

$$\langle O \rangle = \lim_{\alpha\to\infty} \int_{[U]} D[U]\,\mathcal{D}_\alpha[U]\, e^{S_W[U]-\alpha V[U]}\,\mathcal{O}[U]$$

- S_W is the Wilson gauge action (or any other)
- V is the Morse potential (α is an inverse temperature $\alpha \to \infty$)
- $D[U]$ is the Haar measure of $SU(n)$ (invariant under variation)
- $\mathcal{D}_\alpha$ is the gauge-invariant functional ($\to$ FP-trick)

Next, definition of gluon DSE in lattice Landau gauge YM theory

Remember in continuum: $\delta_{\mu\nu}\delta^{ab}\delta(x-y) = \left\langle A^a_\mu(x)\frac{\delta S}{\delta A^b_\nu(y)}\right\rangle$

The gluon Dyson-Schwinger equation on the lattice

Implicit form

$$0 = \lim_{\alpha\to\infty} \int_{[U]} D[U]\; \underline{\boldsymbol{\delta}}^a_{lm} \left(\mathcal{D}_\alpha[U]\, e^{S_W[U]-\alpha V[U]}\, U_{rs} \right)$$

Performing all variations: $\quad 0 = \langle \underline{\boldsymbol{\delta}}^a_{lm} U_{rs} \rangle + \left\langle U_{rs}\, \underline{\boldsymbol{\delta}}^a_{lm} \left(\mathcal{D}_\alpha[U]\, e^{S_W[U]-\alpha V[U]} \right) \right\rangle$

Explicit form

$$\underbrace{\frac{n^2-1}{2n} \langle \mathrm{Tr}\, U_{lm} \rangle \left(\delta_{rl}\delta_{sm} - \delta_{rm}\delta_{sl} \right)}_{\text{variation of } U_{rs}} = \underbrace{\mathfrak{Re} \sum_a \langle K^a_{lm}\, \mathrm{Tr}\, t^a U_{rs} \rangle}_{\text{variation of } 2^{nd} \text{ term}} + \underbrace{\sum_{ab} \left\langle \theta^{a;b}_{r;lm} \mathcal{U}^{ba}_{rs} - \theta^{a;b}_{s;lm} \mathcal{U}^{ba}_{sr} \right\rangle}_{\text{gauge variation of } U_{rs}}$$

- $\langle \cdots \rangle$ is understood as *vev.* over Landau-gauge fixed configurations
- Note: We traced with t^b and exploited global gauge invariance and that the expectation values are real $\to$ *"DSE of gluon field $A_{rs} = A_\nu(y)$"*

The second term on the *rhs* $\quad (=$ longitudinal DSE$)$

Last term on the *rhs* of DSE in momentum space

$$\frac{n^2-1}{2n}\underbrace{\left\langle \frac{1}{N^4}\sum_x \mathrm{Tr}\, U_{x\mu}\right\rangle}_{-V}\delta_{\mu\nu} = \cdots + \sum_{ab,xy} e^{-ip(x-y)}\underbrace{\left\langle \theta^{a;b}_{y;x\mu}\mathcal{U}^{ba}_{y\nu} - \theta^{a;b}_{y+\nu;x\mu}\mathcal{U}^{ab}_{y\nu}\right\rangle}_{L_{\mu\nu}(x-y)}$$

- As above: $\mathcal{U}^{ab}_{y\nu} := \mathfrak{Re}\,\mathrm{Tr}\, t^a t^b U_{y\nu}$ and θ_i is solution of

$$\sum_{c,j} M^{bc}_{ij}\theta^{c;a}_{j;x\mu} = \mathcal{U}^{ba}_{x\mu}(\delta_{ix} - \delta_{ix+\hat{\mu}}) \qquad M = \text{Faddeev-Popov matrix}$$

- $L_{\mu\nu}(x-y)$ is longitudinal
- Gives $\sum_b \left\langle (D^y_\nu c)^b \partial^x_\mu \bar{c}^b_x \right\rangle$ in the continuum limit
- *NOT* the analog of the Kugo-Ojima correlator $\sum_b \left\langle (D^y_\nu c)^b (D^x_\mu \bar{c})^b \right\rangle$
- Longitudinal part of Kugo-Ojima correlator on the lattice (?)

Longitudinal channel of gluon DSE in momentum space

- Longitudinal channel of DSE:

$$-\frac{n^2-1}{2n}\langle V\rangle\,\delta_{\mu\nu} = L_{\mu\nu}(p)$$

- $L_{\mu\nu}(x-y)$ in momentum space

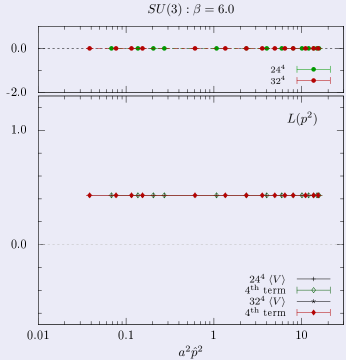
$$L_{\mu\nu}(p) = (\delta_{\mu\nu} - \mathcal{T}_{\mu\nu})L(p)$$

Lattice calculation

- Calculation of $L_{\mu\nu}(p)$ via inversion of FP matrix using plane-wave method
- $SU(3)$ for $\beta = 6.0$ on $N^4 = 24^4, 32^4$

$\Rightarrow$ Longitudinal DSE fulfilled

Long. channel of gluon DSE



Reminder: transverse projector: $\mathcal{T}_{\mu\nu} := \left(\delta_{\mu\nu} - \frac{\hat{p}_\mu\hat{p}_\nu}{\hat{p}^2}\right)$, $\hat{p}_\mu := 2\sin(p_\mu/2)$ and $p_\mu = 2\pi k_\mu/N_\mu$ with $k_\mu = (-N_\mu/2+1, N_\mu/2] \in \mathbb{Z}$

Transverse term of the gluon DSE

First term on the rhs of the DSE in momentum space

$$-\frac{n^2-1}{2n}\langle V\rangle\,\delta_{\mu\nu} = \sum_{xy} e^{-ip(x-y)}\,\mathfrak{Re}\sum_a \left\langle K^a_{x\mu}\ \mathrm{Tr}\ t^a U_{y\nu}\right\rangle + \cdots$$

- Transverse and conserved: $\sum_m K^a_{lm} = 0$
- Continuum: $K^a_\mu(x) = \partial_\nu F^a_{\nu\mu}(x) + j^a_\mu(x)$

  $=$ conserved color Noether current, j^a_μ, plus topologically conserved contribution

where $K^a_{x\mu} = \Sigma^a_{x\mu} + \Phi^a_{x\mu}$ with

$$\Sigma^a_{x\mu} = \frac{2}{g_0^2}\mathfrak{Re}\,\mathrm{Tr}\ t^a U_{x\mu}\sum_{\mu\neq\nu} Staple_{x,\mu\nu}$$

which results from the variation of the Wilson gauge action and the highly non-trivial term $\Phi^a_{x\mu}$ which is due to the variation of $\mathcal{D}_\alpha[U]\,e^{-\alpha V[U]}$

$$\Phi^a_{x\mu} := 2\alpha\sum_i \theta^a_i f^a_i[U] + 2\alpha D_\alpha[U]\int \mathfrak{Re}\,\mathrm{Tr}\,[t^a U_{x\mu}(\mathbb{1} - g^\dagger_{x+\hat\mu} g_x)]e^{-\alpha V[U^g]}\prod_i dg_i$$

Variation of $\mathcal{D}_\alpha[U]\, e^{-\alpha V[U]}$

Remember

$$\mathcal{D}_\alpha^{-1}[U] \xrightarrow{\alpha\to\infty} \sum_k e^{-\alpha V[U_k]} \left(\det M[U_k]\right)^{-1/2}$$

Taking a single (random) Gribov copy U the sum can be expressed by

$$\mathcal{D}_\alpha^{-1}[U] \xrightarrow{\alpha\to\infty} \left(\det M[U]\right)^{-1/2} e^{-(\alpha+\delta\alpha)V[U]}$$

$e^{\delta\alpha V[U]}$ is there because we expect the number of minima to grow exponentially

For $\Phi^a_{x\mu}$ one then finds (in the limit $\alpha \to \infty$)

$$\overline{\Phi}^a_{x\mu} + \delta\alpha \cdot \mathfrak{Re}\,\mathrm{Tr}\, t^a U_{x\mu}$$

where

$$\overline{\Phi}^a_{x\mu} := \frac{1}{2}\frac{\boldsymbol{\delta}^a_{x\mu}(\det M[U])}{\det M[U]} = \frac{1}{2}\sum_{ij,bc} M^{-1\,bc}_{ij}(\delta_{i,x+\hat\mu} - \delta_{ix})\mathfrak{Re}\,\mathrm{Tr}(t^b t^c \delta_{jx} - t^c t^b \delta_{j,x+\hat\mu})t^a U_{x\mu}$$

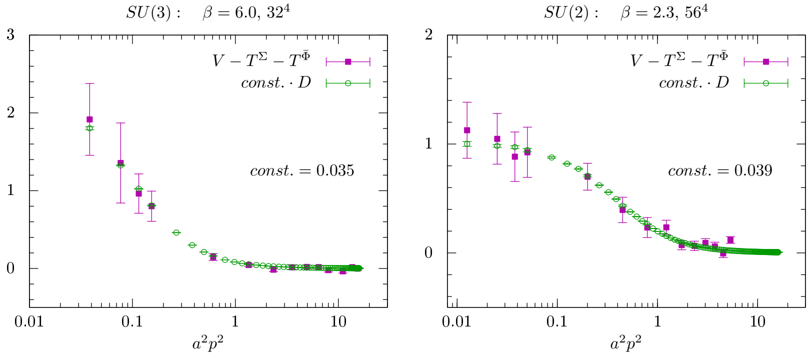
For the evaluation we use the stochastic noise technique for M^{-1} (Gaussian noise)

Transverse term of the gluon DSE

In momentum space the transverse channel of the DSE reads

$$-\frac{n^2-1}{2n}\langle V\rangle = T^{\Sigma}(p) + T^{\Phi}(p) + \delta\alpha D(p)$$

Deviation of $T^{\Sigma}(p) + T^{\Phi}(p)$ from lhs of DSE is proportional to gluon propagator



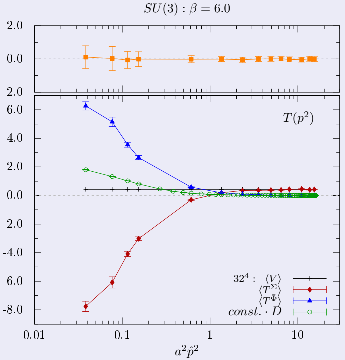
Transverse channel of gluon DSE

$$-\frac{n^2-1}{2n}\langle V\rangle = T^{\Sigma}(p) + T^{\Phi}(p) + \delta\alpha D(p)$$

- $SU(3)$ for $\beta = 6.0$ on $N^4 = 32^4$
- Gluon propagator scaled with $\delta\alpha$

$\Rightarrow$ Transverse DSE fulfilled

Transverse DSE



Summary

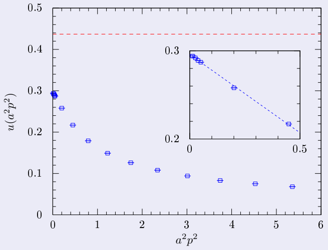
- Have derived (first time) the Dyson-Schwinger equation of a link variable in lattice Landau gauge

$$\frac{n^2-1}{2n}\langle \mathrm{Tr}\, U_{lm}\rangle(\delta_{rl}\delta_{sm}-\delta_{rm}\delta_{sl}) = \mathfrak{Re}\sum_a \langle K^a_{lm}\,\mathrm{Tr}\, t^a U_{rs}\rangle + \sum_{ab}\left\langle \theta^{a;b}_{r;lm}\mathcal{U}^{ba}_{rs} - \theta^{a;b}_{s;lm}\mathcal{U}^{ba}_{sr}\right\rangle$$

- Longitudinal and transverse equation fully satisfied

- For transverse equation have to incorporate corrections due to an insufficient sampling over Gribov copies ($\propto$ gluon propagator)
- First results for the KO-function (see right figure)
- Momentum dependence as expected but the limit $u(p^2 \to 0)$ does not (yet?) satisfy *lhs* of DSE
- Gap at $p = 0$ a bug or feature? Will see!

KO function (preliminary)



Thank you for your attention!